

Tensor glueball–meson mixing phenomenology

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Abstract. The overpopulated isoscalar tensor states are sifted using Schwinger–type mass relations. Two solutions are found: one where the glueball is the $f_J(2220)$, and one where the glueball is more distributed, with $f_2(1820)$ having the largest component. The $f_2(1565)$ and $f_J(1710)$ cannot be accommodated as glueball–(hybrid) meson mixtures in the absence of significant coupling to decay channels. $f'_2(1525) \rightarrow \pi\pi$ is in agreement with experiment. The $f_J(2220)$ decays neither flavour democratically nor is narrow.

1 Introduction

The amount of isoscalar tensor states claimed to exist experimentally [1] has reached a point where naïve interpretation of these states becomes perilous. This is particularly distressing in light of the fact that the tensor glueball, a degree of freedom beyond conventional mesons, is the second lowest glueball predicted by lattice QCD [2–4] and should be manifested in this multiplicity of states. There is a need to bring some order by sorting out well understood states. What is sorely needed is a model–independent theoretical tool.

The need for model–independence of the theory is especially prevalent in light of the fact that the recently claimed $a_2(1660)$ [5–8] has a mass which confounds traditional, and often reliable, potential models of radially excited P–wave mesons. For example, the difference between the first radially excited and ground state isovector $J^{PC} = 2^{++}$ (tensor) states were predicted to be 510 MeV in a relativized quark model [9], while the experimental value is ~ 340 MeV.

In this paper we present a mass–matrix analysis of considerable, although not total, generality. Schwinger–type mass formulae [10] are derived. It is assumed that the mesons and glueball mix only via meson–glueball coupling, with no direct meson–meson coupling. At any stage of the analysis we restrict ourselves to a finite number of mesons (with one glueball) and mixing with hypothetical four–quark states is not taken into account.

The term isoscalar “mesons” shall refer to the partners of the light quark isovector mesons, each of which has an $s\bar{s}$ partner. The isovector mesons will be given by labels like P–wave, F–wave or hybrid meson, indicating the dominant component in a quark model interpretation of the state. However, the mass matrix analysis does not assume the

P–wave, F–wave or hybrid meson nature of a state, nor that it is an unmixed quark model state.

2 Masses

The tensor sector has a few salient features which simplify the analysis. The first excited lattice QCD tensor glueball is 1.85 ± 0.20 times the mass of the scalar glueball [2], and its effect on the experimental spectrum can hence safely be neglected. This means that we can restrict consideration to the low–lying mesons and one primitive (bare) glueball. The mass of the glueball is reliably estimated by using $M(2^{++})/M(0^{++}) = 1.39 \pm 0.04$ [3] or 1.42 ± 0.06 [2], in combination with the average lattice QCD value $M(0^{++}) \approx 1.6$ GeV [11], to be around 2.2 GeV [4]. It is instructive to obtain the lower limit on the tensor glueball mass allowed by lattice QCD. With 1.5 GeV the lower limit for the scalar glueball mass, one obtains a tensor glueball mass $\gtrsim 1.35 M(0^{++}) \gtrsim 2.0$ GeV. This limit will be employed later on.

The isovector tensor mesons should act as beacons for the mass scales of various nonets. Unfortunately, only the $a_2(1320)$, which we will take to fix the mass of the primitive $n\bar{n}$ ground state P–wave state, 1P, is well established [1]. There is recent evidence for $a_2(1660)$ at 1660 ± 40 MeV or 1660 ± 15 MeV [5], for an $a_2(1600 - 1700)$ [6], evidence at ARGUS for a mainly 2^{++} state at 1.7 GeV [7] and for an a_2 at $1752 \pm 21 \pm 4$ MeV [8]. The $a_2(1660)$ is taken to fix the mass of the primitive $n\bar{n}$ first radially excited P–wave state, 2P. Additional evidence for the presence of a 2P nonet is provided by its 1^{++} partners. The $a_1(1700)$ was claimed by BNL [12] and a similar signal was seen at VES [6, 13]. Recently, weak evidence for $f_1(\sim 1700)$ was reported [14].

There is also some recent evidence for isovector tensor states at 2060 ± 20 MeV and 1990_{-30}^{+15} MeV [15], signalling the 3P and 1F nonets. Except for these isovector tensor states, the reasons for expecting the 3P and 1F nonets in

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a similar mass region are as follows. There are $f_0(2010)$, $f_0(2060)$ [1], and $f_J(2100)$, with J most likely 0, at $2115 \pm 15 \pm 15$ MeV [5]. There are recent indications of an a_0 at 2025 ± 30 MeV [15]. These $J = 0$ states signal P-wave mesons, since neither the ground state nor the excited scalar glueball is expected in this mass region [3]. An a_1 at 2100 ± 20 MeV also indicates 3P. Given the experimental mass splitting between the 2P and 1P nonets noted earlier one also expects the 3P level in this mass region. We shall take the primitive $n\bar{n}$ 3P level to be at 2.05 GeV. The 1F nonet is signposted by the $a_4(2050)$, $f_4(2050)$, $K_4(2045)$, the recently reported a_3 at 1860 ± 20 MeV [6] or 2070 ± 20 MeV [15] and f_3 at 2000 ± 40 MeV [14] or 1950 ± 15 MeV [15]. There are recent indications from VES [6] that the mass of the $a_4(2050)$ is $1944 \pm 8 \pm 50$ or 1950 ± 20 MeV. We place the primitive $n\bar{n}$ 1F state at 1.94 GeV and the primitive $s\bar{s}$ state higher by twice the difference between the $K_4(2045)$ and VES' $a_4(2050)$ masses, i.e. at 2.15 GeV. Variation of these masses is discussed in Appendix A.

There is some evidence for an isovector tensor state at 2265 ± 20 MeV [15], signalling the 2F or 4P nonets. The presence of both nonets is indicated by an f_1 at 2340 ± 40 MeV [14], an a_1 at 2340 ± 40 MeV [15], and an f_0 at 2335 ± 25 MeV [15], which can be 4P but not 2F; or $f_4(2300)$, $K_3(2320)$ [1], an a_4 at 2300 ± 20 MeV [15], an f_3 at 2280 ± 30 MeV [14], and a_3 at 2310 ± 40 MeV, which can be 2F but not 4P¹.

It is clear that there is no evidence for overpopulation of levels for isovector tensors, implying that there is no need to introduce a hybrid meson level² up to ~ 2.3 GeV. For isodoublet tensors, there is in fact an underpopulation: only the well-established $K_2^*(1430)$ from the 1P nonet, and the marginal $K_2^*(1980)$ are known [1].

To the contrary, there are 13 isoscalar tensor mesons up to ~ 2.3 GeV listed by the Particle Data Group, with 6 well-established³ [1]. One expects a glueball, and the 1P, 2P, 3P and 1F nonets in this mass region, yielding 9 states, and possibly $n\bar{n}$ 4P and 2F in addition, giving 11 states. There is hence an overpopulation of experimental isoscalar tensors, albeit not for the well-established ones.

Since the 1P, 2P, 3P and 1F nonets are expected below ~ 2.3 GeV, our analysis can safely be restricted to a

¹ $f_4(2300)$ and a_4 may be members of the 1H nonet, although the nonet appears to be more high-lying, as signalled by the $a_6(2450)$ and $f_6(2510)$ [1].

² The possibility of a tensor hybrid meson in the mass range up to ~ 2.3 GeV cannot be excluded theoretically. Beyond the early MIT bag model estimates, constituent gluon models have estimated a tensor hybrid mass, most recently at 1.6–1.8 GeV [40]. Lattice QCD splittings of hybrid levels indicate that at least for $b\bar{b}$ hybrids, the tensor hybrid is degenerate with the lightest hybrids within errors [41]. However, adiabatic lattice QCD and flux-tube models do not find tensors on the lowest hybrid adiabatic surface. Also, tensor mesons are associated with 0^{++} hybrids in bag, constituent gluon and flux-tube models and adiabatic lattice QCD. There is no indication of an overabundance of isovector scalar states.

³ Taking both $f_J(1710)$ and $f_J(2220)$ to have 2^{++} components.

9×9 mass matrix. There is the possibility of the 4P and 2F mesons contaminating results at the upper end of our simulation, at ~ 2.3 GeV, which is also investigated.

3.5 \times 5 mass matrices

The mixing of a glueball and n pairs of isoscalar mesons is described by the following mass matrix, motivated in Appendix B, which is diagonalized by the masses of $(2n + 1)$ physical states:

$$\begin{pmatrix} G & g_1 & g_1\sqrt{2} & g_2 & g_2\sqrt{2} & \cdots & g_n & g_n\sqrt{2} \\ g_1 & S_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ g_1\sqrt{2} & 0 & N_1 & 0 & 0 & \cdots & 0 & 0 \\ g_2 & 0 & 0 & S_2 & 0 & \cdots & 0 & 0 \\ g_2\sqrt{2} & 0 & 0 & 0 & N_2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ g_n & 0 & 0 & 0 & 0 & \cdots & S_n & 0 \\ g_n\sqrt{2} & 0 & 0 & 0 & 0 & \cdots & 0 & N_n \end{pmatrix} \quad (1)$$

$$\implies \text{diag}(h_1, h_2, h_3, \dots, h_{2n}, h_{2n+1}).$$

G and S, N stand for the mass of the primitive glueball, and $s\bar{s}$ and $n\bar{n} \equiv (u\bar{u} + d\bar{d})/\sqrt{2}$ mesons, respectively, the subscript indicating the number of the nonet the state belongs to. h_i stand for the masses of the physical states. g_i are the glueball–meson couplings that have dimensionality (mass), in accord with the dimensionality of the diagonal entries of (1). In what follows, we restrict ourselves to the case where the quantities in (1) are real numbers⁴.

Applying the techniques of [10], one can obtain $2n$ pairs of relations for the coupling in terms of the primitive and physical masses:

$$g_i = \sqrt{-\frac{\prod_{j=1}^{2n+1} (S_i - h_j)}{\prod_{j=1}^n (S_i - N_j) \prod_{j=1, j \neq i}^n (S_i - S_j)}}, \quad i = 1, 2, \dots, n,$$

$$g_i = \sqrt{-\frac{\prod_{j=1}^{2n+1} (N_i - h_j)}{2 \prod_{j=1}^n (N_i - S_j) \prod_{j=1, j \neq i}^n (N_i - N_j)}}, \quad (2)$$

Each pair of these relations represents a Schwinger-type mass formula. Hence, for $(2n + 1) \times (2n + 1)$ mass matrix (1) one has n Schwinger mass relations. These n formulae, together with the trace condition for the mass matrix (1),

$$G + S_1 + N_1 + S_2 + N_2 + \dots + S_n + N_n = h_1 + h_2 + h_3 + \dots + h_{2n} + h_{2n+1}, \quad (3)$$

constitute $n + 1$ mass relations for the mixing of a glueball and n meson nonets. It is clear that solving such a

⁴ g_i and $-g_i$ gives the same eigenvalues, so we always choose g_i non-negative.

system of $n + 1$ mass relations can lead to unphysical solutions, e.g., solutions that correspond to all or some of the couplings being imaginary numbers. Obviously, such solutions will not correspond to the initial mass matrix (1), and hence should be rejected.

It is difficult to find all the solutions of the Schwinger equations for a 9×9 mass matrix of the form (1) numerically. Hence we take the approach to solve the Schwinger equations for a 5×5 mass sub–matrix, which involves only the primitive and physical masses, and then reconstruct the couplings. As all 5×5 sub–matrices are found, we then obtain the corresponding 9×9 mass matrix.

In this analysis, we pursue the following strategy:

(i) We start with the 5×5 sub–matrix for the glueball and 2P and 1F nonets, by fixing the masses of primitive $n\bar{n}$ for 2P, and $n\bar{n}$ and $s\bar{s}$ for 1F, and obtain the primitive 2P $s\bar{s}$ mass. We then consider another 5×5 sub–matrix for the glueball and 2P and 3P nonets, with fixed: both $n\bar{n}$ and $s\bar{s}$ masses of the 2P nonet from the previous simulation, and the primitive $n\bar{n}$ mass for the 3P nonet. In both 5×5 sub–matrix simulations we obtain *all* the solutions of the Schwinger equations (2).

(ii) *Input:* For the two 5×5 sub–matrix simulations, we fix the following values of the primitive masses (in GeV): $N = 1.66$ for 2P, $N = 1.94$, $S = 2.15$ for 1F and $N = 2.05$ for 3P. We also take one of the physical states to have a mass in agreement with one of the glueball candidates (which we review in the next section), and the other three physical states to have masses in agreement with three states among $f_2(1565)$, $f_2(1640)$, $f_J(1710)$, $f_2(1810)$, $f_2(1950)$, $f_2(2010)$, $f_2(2150)$, $f_J(2220)$, $f_2(2300)$ and $f_2(2340)$, excluding the state already chosen for the physical glueball. *Output:* We then solve the system of three equations (two Schwinger formulae and the trace condition) for three unknowns: G , S for 2P and the remaining fifth physical mass for the first 5×5 simulation, and G , S for 3P and the remaining fifth physical mass for the second 5×5 simulation. We require that the fifth physical mass from each 5×5 simulation is among the physical states mentioned above.

(iii) Since we take the $f_2(1275)$ and $f_2'(1525)$ as the established ground state 1P tensor mesons, we incorporate them later in the full 9×9 mass matrix analysis.

(iv) We discard the possibility that $f_2(1420)$ exists. Although claimed by a number of old experiments in a variety of production processes, recent experiments do not confirm its existence. This is most vividly illustrated by its observation in (mostly) double Pomeron exchange in $pp \rightarrow p_f(\pi^+\pi^-)p_s$ at $\sqrt{s} = 63$ GeV [4, 16]. Recent examination of the *same* reaction does not see any evidence for $f_2(1420)$ [17].

We admit the following criteria for holding physical solutions and separating out non–physical ones:

(i) The output fifth physical mass lies within a mass range allowed by data for one of the experimental candidates.

(ii) The mass of the primitive glueball satisfies $G \geq 2$ GeV.

(iii) In all the cases when a primitive $s\bar{s}$ mass is to be obtained, it is higher than the corresponding $n\bar{n}$ mass, and the $s\bar{s} - n\bar{n}$ mass splitting is consistent with the quark model motivated estimate 200 ± 50 MeV [11].

4 9×9 mass matrices

Various physical states have been suggested as tensor glueball candidates in the literature:

$f_J(2220)$: The $f_J(2220)$ is strongly produced in J/ψ radiative decay, and not seen in $\gamma\gamma$ collisions, suggesting glueball character if $J = 2$ [4, 18]. The flavour democratic decay pattern and small total width of $f_J(2220)$ is also cited as evidence for its glueball nature [19].

$f_2(2150)$: This was suggested in [20].

The nearness of the mass of $f_J(2220)$ and $f_2(2150)$ to the tensor glueball mass predicted by lattice QCD is often cited as evidence for their glueball nature [4].

$f_2(1950)$: The p_T dependence of the pp central production of $f_2(1950)$ is consistent with its glueball character according the Close–Kirk glueball filter [4, 21]. However, it was admitted that the structure seen in central production may represent more than one resonance [21].

$f_J(1710)$: The glueball nature of this state is suggested by its p_T dependence in central production [21] and its production in “glue–rich” $p\bar{p}$ annihilation [18], although its strong production in J/ψ radiative decay is consistent with expectations for $q\bar{q}$ if $J=2$ [18].

For each 5×5 case, we take one of the above four glueball candidates to be one of the five physical states.

Having completed the double 5×5 sub–matrix analysis and fixing $N = 1.318$ for 1P [1], the full 9×9 mass matrix is now recovered by solving the Schwinger equations (2) exactly, using the solutions obtained for the two 5×5 matrices as initial values for the search routine. We therefore do *not* obtain *all* of the 9×9 mass matrix solutions, but only the ones similar to the ones found formerly with the two 5×5 matrices.

There are two different solutions, which are almost identical with respect to the physical masses. Particularly, the primitive glueball masses are consistent with 2.0 – 2.1 GeV predicted by recent models [4, 22] and the lattice QCD predictions mentioned earlier. The couplings are in the range 30 – 120 MeV for the various nonets. These values are similar to 43 ± 31 MeV predicted in lattice QCD for ground state isoscalar scalars [23]. We find that the physical masses are insensitive to changes in the input, but that the valence content is more sensitive: especially for states at similar masses to where the parameters are changed, and for small valence components (see Appendix A).

(i) For the first solution, the $f_J(2220)$ turns out to be the physical glueball. For this solution, the initial 9×9 mass matrix is (shown are the values of the primitive masses and couplings rounded to the second decimal digit;

all values are given in GeV)

$$\begin{pmatrix} 2.10 & 0.03 & 0.03\sqrt{2} & 0.04 & 0.04\sqrt{2} & 0.09 & 0.09\sqrt{2} & 0.12 & 0.12\sqrt{2} \\ 0.03 & 2.28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.03\sqrt{2} & 0 & 2.05 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.04 & 0 & 0 & 2.15 & 0 & 0 & 0 & 0 & 0 \\ 0.04\sqrt{2} & 0 & 0 & 0 & 1.94 & 0 & 0 & 0 & 0 \\ 0.09 & 0 & 0 & 0 & 0 & 1.84 & 0 & 0 & 0 \\ 0.09\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 1.66 & 0 & 0 \\ 0.12 & 0 & 0 & 0 & 0 & 0 & 0 & 1.55 & 0 \\ 0.12\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.318 \end{pmatrix}, \quad (4)$$

the physical masses are

$$2.29, 2.23, 2.14, 2.04, 1.93, 1.82, 1.64, 1.52, 1.28, \quad (5)$$

and the valence content of the physical states is

$$\begin{pmatrix} \underline{0.38} & \mathbf{0.91} & \underline{0.07} & \underline{0.11} & \underline{0.06} & \underline{0.08} & \underline{0.08} & \underline{0.06} & \underline{0.07} \\ \underline{0.73} & \underline{-0.42} & \underline{0.18} & \underline{0.38} & \underline{0.14} & \underline{0.17} & \underline{0.17} & \underline{0.13} & \underline{0.14} \\ \underline{-0.32} & \underline{0.07} & \underline{-0.16} & \mathbf{0.92} & \underline{-0.09} & \underline{-0.10} & \underline{-0.08} & \underline{-0.07} & \underline{-0.07} \\ \underline{-0.18} & \underline{0.02} & \mathbf{0.97} & \underline{0.07} & \underline{-0.10} & \underline{-0.08} & \underline{-0.06} & \underline{-0.05} & \underline{-0.04} \\ \underline{-0.15} & \underline{0.01} & \underline{0.05} & \underline{0.03} & \mathbf{0.97} & \underline{-0.14} & \underline{-0.07} & \underline{-0.05} & \underline{-0.04} \\ \underline{-0.19} & \underline{0.01} & \underline{0.04} & \underline{0.02} & \underline{0.10} & \mathbf{0.96} & \underline{-0.15} & \underline{-0.09} & \underline{-0.07} \\ \underline{-0.18} & \underline{0.01} & \underline{0.02} & \underline{0.01} & \underline{0.03} & \underline{0.08} & \mathbf{0.94} & \underline{-0.25} & \underline{-0.10} \\ \underline{-0.20} & \underline{0.01} & \underline{0.02} & \underline{0.01} & \underline{0.03} & \underline{0.06} & \underline{0.19} & \mathbf{0.94} & \underline{-0.17} \\ \underline{-0.23} & \underline{0.01} & \underline{0.01} & \underline{0.01} & \underline{0.02} & \underline{0.04} & \underline{0.08} & \underline{0.10} & \mathbf{0.96} \end{pmatrix}. \quad (6)$$

(ii) For the second solution, the the physical glueball is distributed, with $f_2(1810)$ containing the largest component. Although the highest mass state appears to have the largest glueball component, we shall see in Sect. 6 that the content changes as more high mass states are introduced. For this solution, the initial 9×9 mass matrix is

$$\begin{pmatrix} 2.05 & 0.10 & 0.10\sqrt{2} & 0.11 & 0.11\sqrt{2} & 0.08 & 0.08\sqrt{2} & 0.11 & 0.11\sqrt{2} \\ 0.10 & 2.27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.10\sqrt{2} & 0 & 2.05 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.11 & 0 & 0 & 2.15 & 0 & 0 & 0 & 0 & 0 \\ 0.11\sqrt{2} & 0 & 0 & 0 & 1.94 & 0 & 0 & 0 & 0 \\ 0.08 & 0 & 0 & 0 & 0 & 1.95 & 0 & 0 & 0 \\ 0.08\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 1.66 & 0 & 0 \\ 0.11 & 0 & 0 & 0 & 0 & 0 & 0 & 1.55 & 0 \\ 0.11\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.318 \end{pmatrix}, \quad (7)$$

the physical masses are

$$2.38, 2.23, 2.12, 2.01, 1.95, 1.82, 1.63, 1.52, 1.28, \quad (8)$$

and the valence content of the physical states is

$$\begin{pmatrix} \underline{0.64} & \underline{0.58} & \underline{0.27} & \underline{0.31} & \underline{0.23} & \underline{0.12} & \underline{0.10} & \underline{0.08} & \underline{0.09} \\ \underline{0.31} & \underline{-0.79} & \underline{0.24} & \underline{0.42} & \underline{0.17} & \underline{0.09} & \underline{0.06} & \underline{0.05} & \underline{0.05} \\ \underline{-0.22} & \underline{0.15} & \underline{-0.45} & \underline{0.82} & \underline{-0.19} & \underline{-0.11} & \underline{-0.06} & \underline{-0.04} & \underline{-0.04} \\ \underline{-0.22} & \underline{0.08} & \underline{0.75} & \underline{0.17} & \underline{-0.50} & \underline{-0.30} & \underline{-0.07} & \underline{-0.05} & \underline{-0.05} \\ \underline{-0.02} & \underline{0.01} & \underline{0.03} & \underline{0.01} & \underline{-0.47} & \underline{0.88} & \underline{-0.01} & \underline{-0.01} & \underline{-0.01} \\ \underline{-0.48} & \underline{0.11} & \underline{0.30} & \underline{0.16} & \underline{0.63} & \underline{0.30} & \underline{-0.33} & \underline{-0.19} & \underline{-0.15} \\ \underline{-0.22} & \underline{0.04} & \underline{0.08} & \underline{0.05} & \underline{0.11} & \underline{0.06} & \mathbf{0.91} & \underline{-0.30} & \underline{-0.11} \\ \underline{-0.25} & \underline{0.03} & \underline{0.07} & \underline{0.04} & \underline{0.09} & \underline{0.07} & \underline{0.20} & \mathbf{0.92} & \underline{-0.19} \\ \underline{-0.25} & \underline{0.03} & \underline{0.05} & \underline{0.03} & \underline{0.06} & \underline{0.03} & \underline{0.07} & \underline{0.10} & \mathbf{0.96} \end{pmatrix}. \quad (9)$$

5 Discussion of mass matrix results

Both solutions have the following similarities:

(i) The physical states are $f_2(1270)$, $f_2'(1525)$, $f_2(1640)$, $f_2(1810)$, $f_2(1950)$, $f_2(2010)$, $f_2(2150)$, $f_J(2220)$ and either the $f_2(2300)/f_2(2340)$ or $f_2(2340)$ as solutions in the first and second cases, respectively⁵. We never found the $f_2(1565)$ and $f_J(1710)$. The reason why $f_2(1810)$ is found instead of these resonances is because the primitive $2P \bar{s}\bar{s}$ is required to 250 ± 50 MeV from the input $a_2(1660)$ mass.

(ii) The valence content has almost entirely the same signs between the various components, the only exception being different signs for the two dominant meson components in $f_2(1950)$.

(iii) The physical mesons have a substantial glueball content, contrary to naïve expectations, with the exception of $f_2(1950)$ in solution 2. It has been argued phenomenologically that experimental data demand physical mesons with appreciable glueball content [24]. This would, for example, explain why $f_2(2010)$, $f_2(2300)$ and $f_2(2340)$ were observed in the OZI forbidden process $\pi p \rightarrow \phi \phi n$, and would suggest that several tensor mesons should be produced in glue-rich processes. For example, $f_2(1270)$ was observed in gluon fusion [25]. The small glueball component in $f_2(1950)$ in solution 2 is apparently in contradiction with the Close–Kirk filter.

(iv) The $f_2(1270)$, $f_2'(1525)$ and $f_2(1640)$ are composed of more than 90% of the expected primitive state.

(v) For the 1P nonet, $S + N = 1.55 + 1.318 = 2.868$ GeV is consistent with $2M(K_2^*) = 2.858 \pm 0.01$ GeV [1].

The solutions differ as follows:

(i) In the first solution all physical mesons have one component which has a valence content of larger than 90%, i.e. the state is dominantly a specific primitive state. For this solution, valence components of physical mesons greater than 10% only occur within two primitive states of the dominant primitive state. The second solution does not behave in this way.

⁵ Within experimental mass uncertainty [1], we cannot distinguish between $f_2'(1525)$ or $f_2(1565)$. In Sect. 7.1, we show that $f_2(1525) \rightarrow \pi\pi$ is consistent with experiment if it is taken to have the valence content of the second state, which we thus identify as $f_2'(1525)$.

(ii) For the first solution, the physical glueball has substantial valence content in all the meson states it couples to, contrary to the second solution.

(iii) In the first solution the couplings decrease with increased radial excitation (from 1P to 3P), as one would naïvely expect [26]; while for the second solution the couplings remain approximately the same, as expected from Regge theory [27]. An advantage of our method is that all information on couplings are predictions, using (2), once the masses are known.

(iv) The first solution has the slight disadvantage that the primitive $s\bar{s}$ – $n\bar{n}$ mass splitting ~ 180 MeV in the 2P nonet is a little small, versus an agreeable ~ 290 MeV for the second solution.

(v) For the second solution, it is seen in (9) that both the $f_2(1950)$ and $f_2(2150)$ have large $n\bar{n}$ and $s\bar{s}$ components, and neither of them are in destructive interference. Hence, both of these states will have many different decay modes, which is in excellent agreement with data (these different decay modes are amongst the main reasons for each of these states to be chosen as the tensor glueball candidate by different groups). Although this feature is absent for these states for the first solution (4)–(6), where $f_2(1950)$ is mostly $n\bar{n}$ and $f_2(2150)$ mostly $s\bar{s}$, many of the observed decay modes arise from connected decay of both $u\bar{u}$ and $s\bar{s}$ components, so that this avenue to distinguish between solutions may not be definitive.

5.1 $f_2(1565)$ and $f_J(1710)$

It is possible that $f_2(1565)$ and $f_2(1640)$ are aspects of the same state, which would remove one extra state. We have nevertheless attempted to find solutions where $f_2(1565)$ and $f_J(1710)$ are physical states in addition to the states (5),(8), by adding another nonet to form an 11×11 matrix. Of course, one can insert primitive states at these masses with an unrealistically small glueball–meson coupling and an unrealistic $s\bar{s}$ – $u\bar{u}$ mass splitting of 150 MeV and obtain a solution. However, no realistic solutions are found.

We have neglected the mixing of mesons with decay channels throughout, since it is believed to produce only tiny mass shifts [28]. However, scenarios where there is large coupling to decay channels, e.g. $\omega\omega$, $\rho\rho$, K^*K^* and $\phi\phi$, have been advanced by various authors.

$f_2(1565)$ decays to $\rho\rho$ and $\omega\omega$ and has an abnormally small branching ratio to $\pi\pi$ and $\eta\eta$ [1]. This, together with the nearness of $f_2(1565)$ to the $\rho\rho$ and $\omega\omega$ thresholds has lead to suggestions that $f_2(1565)$ is a $\rho\rho$ molecule or a baryonium state [29]. Also, $f_2(1565)$ may be the isoscalar partner of the isotensor tensor enhancement $X(1600)$ [1], which, if it is resonant, must be a degree of freedom beyond glueballs and (hybrid) mesons.

A more modest suggestion is that the mass of the 2P $n\bar{n}$ state found in our formalism is shifted downward by the $\rho\rho$ and $\omega\omega$ thresholds, which the 3P_0 model predicts it to couple strongly to [30].

$f_J(1710)$ has been suggested as a K^*K^* molecule [31]. However, it is not well established that a $J = 2$ component exists. BES separated both $J = 0$ and $J = 2$ components,

with the tensor state having mass 1697 MeV and a width of 176 MeV [32]. However, recent evidence support only the $J = 0$ component [17,33].

6 13×13 mass matrix

Once the 9×9 mass matrix is fixed, one can easily add extra meson nonets to it. We add the primitive $n\bar{n}$ and $s\bar{s}$ masses of the 2F nonet at 2.3 and 2.5 GeV, respectively, to the mass matrices (4),(7). Similarly, the primitive states of the 4P nonet are added at 2.35 and 2.55 GeV, to yield a 13×13 mass matrix. The physical states in (5),(8) are required to be among the physical states, i.e. *both* $f_2(2300)$ and $f_2(2340)$.

The result is a 13×13 “counterpart” to each 9×9 matrix (6),(9), called solutions 1a and 2a. The primitive states in common have the same couplings and primitive masses, and similar valence content (with the same signs). The valence content of a given primitive state tends to decrease from the 9×9 counterpart, since the physical state is spread over more primitive states.

Remarkably, the ratio of valence contents of the 13×13 solutions and their 9×9 counterparts remain extremely similar ($\sim 1\%$), except for the components in the 9×9 matrix which has similar mass to the new components being added. This means that for low-lying states, there is usually no need extend the number of primitive components in order to study decay.

We also find two new solutions, called 1b and 2b, since they are respectively similar to the 9×9 solutions 1 and 2. They have, however, no 9×9 counterparts. Solution 2b is displayed in Appendix A.

In all 13×13 solutions, three new, experimentally undiscovered, physical states appear at masses beyond the $f_2(2340)$. The dominant glueball component in solutions 2a and 2b is found in one of the three new high mass states.

We note that because the number of nonets stable under decay is expected to be finite due to pair creation in QCD [34], the largest mass matrix that need to be analysed is finite.

7 Decays

In order to calculate the decay of a physical state to an exclusive final state it is necessary to add the decay amplitudes of all its primitive components, weighted by their valence content. This is demonstrated in Appendix C.

The decay amplitudes of the primitive components will be calculated in the 3P_0 model, meaning that pair creation is with vacuum quantum numbers and decays proceed via a connected quark diagram. Unless otherwise noted, the decays are calculated using the “relativistic” phase space convention and parameters of [30]. Another convention is “mock meson” phase space with the parameters of [35]. In the mass matrix analysis, specific quark model identifications were not assumed for the various components.

However, to calculate the decays, the quark model content indicated by the label of a component will be assumed.

The preceding mechanism whereby the physical glueball decays via primitive meson components, which is closely related to the primitive glueball decaying via intermediate primitive mesons [27], provides the first theoretical understanding of the scalar glueball decay pattern found in lattice QCD [11].

When one combs the experimental data available on isoscalar tensor mesons more massive than the $f_2'(1525)$, there is very little robust quantitative information available that can directly be compared to theory. The most restrictive datum appears to be the ratio of widths of $f_J(2220)$ to $\pi\pi$ and $K\bar{K}$, which we shall analyse below. Of the qualitative data available, the observation or non-observation of various isoscalar tensor mesons in $\phi\phi$ is of particular interest, as the $\phi\phi$ decay can arise for connected decay only from the $s\bar{s}$ components of states. We shall not make exhaustive decay predictions, but restrict to the cases mentioned in a genuine attempt to confront our picture with experiment. However, we first analyse the OZI forbidden decay $f_2'(1525) \rightarrow \pi\pi$ which is zero in models where the state has only one valence component.

7.1 $f_2'(1525) \rightarrow \pi\pi$

For the 13×13 solutions 1a, 1b, 2a and 2b we find $\Gamma(f_2'(1525) \rightarrow \pi\pi) = 1.6(1.4), 1.2(1.1), 1.0(0.9), 0.7(0.7)$ MeV, with relativistic phase space⁶ listed first. Solution 2b is in best agreement with the experimental value 0.60 ± 0.12 MeV. The two 9×9 solutions gives the same results as their 13×13 counterparts. As seen in (6),(9), the valence content of the $f_2'(1525)$ is such that its 1P and 2P $n\bar{n}$ components are in destructive interference (they have opposite signs), which will result in the suppression of the $\pi\pi$ decay mode of this state. Furthermore, for components higher in mass than 2P, the valence content has the same sign as the 2P component, leading to further suppression. To be specific, we illustrate this for solution 2b. $\Gamma(f_2'(1525) \rightarrow \pi\pi) = 14$ and 4 MeV if only 1P, and 1P and 2P components are included. The width remains above 1.5 MeV as long as not all of the 1P, 2P, 1F and 3P components are included.

We have thus provided the first quantitative understanding of the process $f_2'(1525) \rightarrow \pi\pi$. This demonstrates that the techniques of both the mass matrix and 3P_0 decay analysis yield predictions consistent with experiment, motivating their continued use. Remarkably, the decay $f_2'(1525) \rightarrow \pi\pi$ can only be understood when at least four different components of $f_2'(1525)$ are included. It is also apparent that there is no need to postulate a non-connected decay mechanism, whereby primitive $s\bar{s}$ components would directly decay to $\pi\pi$. Because $f_2'(1525)$ is dominantly $s\bar{s}$, such processes must be small indeed.

⁶ In accordance with [30] we use a slightly higher pair creation constant for low-mass states. For $\Gamma(f_2(1275) \rightarrow \pi\pi)$, with $f_2(1275)$ purely 1P $n\bar{n}$, this gives 160 MeV, in perfect agreement the experimental 157 ± 4 MeV [1].

7.2 $R \equiv \Gamma(f_J(2220) \rightarrow \pi^+\pi^-)/\Gamma(f_J(2220) \rightarrow K^+K^-)$ ($J = 2$)

The main distinguishing characteristic of the $f_J(2220)$ is its remarkably narrow total width of 23_{-7}^{+8} MeV [1] for a state that can decay via numerous decay modes [36]. Not even the existence of $f_2(2220)$ is well-established [4] as the narrow peak sit on a variety of empirical backgrounds, for which there is no explanation [37], so that the peak might be a statistical fluctuation. Moreover, broader states in the same mass region have been reported: JETSET sees an f_2 at 2231 ± 2 MeV with a width of 70 ± 10 MeV [38]. An f_2 at 2240 ± 40 MeV with a width of 170 ± 50 MeV [14], and at 2210 ± 45 MeV with width 260 ± 45 [15] have also been reported. It is possible to understand current data on $f_J(2220)$ if one does not take it to be narrow [37]. If the $f_J(2220)$ is not narrow, none of the “indicators” of its glueball nature can be sustained, for example its non-observation in $\gamma\gamma$ collisions [4] simply follows from its wideness, and its coupling to gluons in J/Ψ radiative decay becomes compatible to conventional mesons if $J = 2$ [18].

As seen in (6),(9), the largest non-gluon components of the $f_J(2220)$, 3P and 1F $s\bar{s}$, are in destructive interference, and remain so in the 13×13 solutions. This also tends to be true for the largest $n\bar{n}$ components in the 13×13 solutions. One may think this will result in the suppression of the decay modes of this state, making its total width consistent with the tiny experimental value. Evaluating the total width of $f_J(2220)$, with $J = 2$, to $\pi\pi$ and $K\bar{K}$, for all the 13×13 solutions, and phase space conventions, we obtain 20 – 150 MeV. It is evident that the tiny total width can not be sustained in our model, and it likely is to be a challenge to any model in which the physical glueball has non-negligible mixing with mesons [36]. The individual partial widths to $\pi\pi$ and $K\bar{K}$ are also inconsistent with experimental bounds that assume a narrow $f_J(2220)$ [4].

It is often claimed that $f_J(2220)$ has a flavour democratic decay pattern expected for a pure glueball [19], whereby $R = 1$ without phase space included, and $R = 1.7$ with phase space included. However, naïve flavour factors give that a pure $n\bar{n}$ and $s\bar{s}$ state should have, without phase space included, $R = 4, 0$ respectively. Thus a mixture between $n\bar{n}$ and $s\bar{s}$ can also look flavour democratic. For the 13×13 solutions 1a, 1b, 2a and 2b we find $R = 0.6, 0.7, 0.4, 0.5$ respectively, independent of phase space conventions. The two 9×9 solutions gives the same results as their 13×13 counterparts. Although these values of R do not represent flavour democratic decay, they are all consistent with experiment [19], which possesses large error bars. This is true for solutions 1a and 1b where $f_J(2220)$ is the physical glueball, and for the other solutions.

When decays are to final S-wave mesons, i.e. $\pi, \eta, K, \rho, \omega, K^*, \eta'$ or ϕ , one almost always finds that the decay amplitudes decrease sharply as the decaying component is progressively radially excited. The same is true as the decaying component is orbitally excited. This has the consequence that although a physical state may have a domi-

Table 1. Decay widths to $\phi\phi$ in MeV for various solutions. Relativistic phase space is used first, with mock meson phase space in brackets. The two 9×9 solutions give almost identical results to their 13×13 counterparts. $f_2(2010)$ is taken to have a mass at the upper end of the experimental range [1]

13×13 solution	$f_2(2010)$	$f_2(2150)$	$f_J(2220)$	$f_2(2300)$	$f_2(2340)$
1a	3(3)	7(5)	17(12)	23(16)	7(4)
1b	3(2)	5(4)	12(9)	17(12)	8(5)
2a	18(15)	3(3)	0.6(0.5)	4(3)	9(6)
2b	12(11)	1.7(1.4)	0.7(0.5)	7(5)	2(2)

nantly excited component, its decay dominantly proceeds through a lower excited component. This means that naïve quark model calculations that assign a single component to an excited state [30,35] might be completely unreliable. One would *a priori* expect this situation to be worst in J^{PC} sectors where low-lying glueballs are present, i.e. $J^{PC} = 0^{++}, 2^{++}$ and 0^{-+} [3]. We illustrate the phenomenon by analysing $f_J(2220) \rightarrow \pi\pi$ and $K\bar{K}$ for the 13×13 solutions. Although $f_J(2220)$ is never dominantly 1P, this contribution is always one of the dominant ones. One tends to find that half of the width can be found by including only the 1P and 2P contributions, even though the state may be dominantly 1F and 3P.

7.3 Decays to $\phi\phi$

We shall analyse the decay of various resonances to $\phi\phi$, in an attempt to understand the data, which claim that $f_2(2010)$, $f_2(2300)$ and $f_2(2340)$ have been seen in $\phi\phi$ in $\pi\pi$ collisions [1]. There is also preliminary evidence for an f_2 at $\sim 2231 \pm 2$ MeV in $\phi\phi$ [38]. We bear in mind that the production process can alter conclusions made based on studying decays.

The results are in Table 1. From phase space considerations, it is especially surprising that it is possible for $f_2(2150)$ to have a smaller width to $\phi\phi$ than $f_2(2010)$. This is even more surprising, given that $f_2(2010)$ is dominantly $n\bar{n}$ and $f_2(2150)$ is dominantly 1F $s\bar{s}$. The resolution of the paradox is that 1F and 2F does not decay to $\phi\phi$ in the dominant S-wave in the 3P_0 model [30]. We note that solutions 2a and 2b are consistent with states unambiguously observed in $\phi\phi$.

8 Salient features

We showed that Schwinger–type mass formulae can be obtained when we restrict to glueball–(hybrid) meson mixing. With some physical isovector and isoscalar masses known, these formulae can predict unknown masses and couplings. The utility of this new analysis technique was demonstrated in the tensor sector.

It has been shown that in order to understand the decay $f_2'(1525) \rightarrow \pi\pi$, one has to consider more than the 1P

$n\bar{n}$ component. This implies that the use of a 2×2 mixing formula where the physical states are linear combinations of $n\bar{n}$ and $s\bar{s}$ components can be inadequate.

In our approach the physical glueball is *a priori* narrower than mesons due to the large glueball component, which is taken not to decay. However, as shown for the $f_J(2220)$, the physical glueball is not unusually narrow, because of the presence of significant meson components, contrary to the perturbative QCD claim that glueball mixing with 1F mesons is tiny [26]. Also, there is no reason to expect a flavour democratic decay pattern for the physical glueball. It is a common myth that glueballs are narrow and decay flavour democratically. Experimentally, this is already clear for the scalar glueball [4,11]. These notions arise from perturbative QCD, which, paradoxically, has been argued to be valid for the tensor glueball but not for the scalar glueball [39]. However, glueballs beyond 3 GeV can be narrow [34]. A more reliable glueball signature may be glue–rich production.

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References

1. Particle Data Group (C. Caso et al.) Eur. Phys. J. **C3** (1998) 1.
2. M.J. Teper, hep-th/9812187.
3. C.J. Morningstar, M. Peardon, hep-lat/9901004; Phys. Rev. **D56** (1997) 4043; M. Peardon, Nucl. Phys. (Proc. Suppl.) **B63** (1998) 22.
4. S. Godfrey and J. Napolitano, hep-ph/9811410, submitted to Rev. Mod. Phys.
5. A. Abele et al. (CBAR Collab.) Eur. Phys. J. **C8** (1999) 67; A.V. Anisovich et al., Phys. Lett. **B449** (1999) 145.
6. D.V. Amelin et al. (VES Collab.) hep-ex/9810013; hep-ex/9905002.
7. G. Kernel, *Proc. of PHOTON'95*, p. 226, eds. D.J. Miller et al.; E. Križnič, *Proc. of XXVII Int. Conf. of High Energy Physics (ICHEP'94)* (1994, Glasgow, U.K.) p. 1413, eds. P. Bussey, I. Knowles (IOP, London, 1995).
8. S.R. Hou (L3 Collab.) *Proc. of 7th Int. Conf. on Hadron Spectroscopy (HADRON'97)* (Aug. 1997, Upton, NY) p. 745, eds. S.-U. Chung, H.J. Willutzki, AIP conference proceedings 432 (AIP, New York, 1998).
9. S. Godfrey, N. Isgur, Phys. Rev. **D32** (1985) 189.
10. L. Burakovsky, T. Goldman, P.R. Page, hep-ph/9812395.
11. L. Burakovsky, P.R. Page, Phys. Rev. **D59** (1999) 014022; Erratum *ibid.* **D59** (1999) 079902; and references therein.
12. J.H. Lee et al. (E818 Collab.) Phys. Lett. **B323** (1994) 227.
13. D.V. Amelin et al. (VES Collab.) Phys. Lett. **B356** (1995) 595; A.M. Zaitsev, *Proc. of ICHEP'94*, p. 1409 [7].
14. A.V. Anisovich et al., “Study of $p\bar{p} \rightarrow \pi^0\pi^0\eta$ from 600 to 1940 MeV/c”, submitted to Phys. Lett. B (1999); B.-S. Zou, *Proc. of Low Energy Antiproton Physics Conference (LEAP'98)* (Sep. 1998, Villasimius, Italy) hep-ex/9812007.
15. A.V. Sarantsev, D.V. Bugg, “I=0 $q\bar{q}$ resonances from 1.9 to 2.4 GeV”; “I=1 $q\bar{q}$ resonances in the mass range 2.0–2.4

- GeV^m, *Proc. of Workshop on Hadron Spectroscopy* (March 1999, Frascati, Italy).
16. T. Akesson et al., *Nucl. Phys.* **B264** (1986) 154.
 17. D. Barberis et al. (WA102 Collab.) hep-ex/9903042; hep-ex/9903043.
 18. F.E. Close, G. Farrar, Z.-P. Li, *Phys. Rev.* **D55** (1997) 5749; F.E. Close, *Nucl. Phys. Proc. Suppl.* **56A** (1997) 248.
 19. J.Z. Bai (BES Collab.) *Phys. Rev. Lett.* **76** (1996) 3502; X. Shen, *Proc. of HADRON'97*, p. 47 [8].
 20. Yu.D. Prokoshkin (GAMS Collab.) *Phys. Dokl.* **40** (1995) 495.
 21. A. Kirk (WA102 Collab.) *Proc. of HADRON'97*, p. 715 [8], hep-ex/9709024.
 22. A.P. Szczepaniak, E.S. Swanson, C.-R. Ji, S.R. Cotanch, *Phys. Rev. Lett.* **76** (1996) 2011.
 23. W. Lee and D. Weingarten, hep-ph/9805029; *Proc. of 6th Int. Symp. on Lattice Field Theory (LATTICE'98)* (July 1998, Boulder, CO) hep-lat/9811005.
 24. D.V. Bugg, B.-S. Zou, *Phys. Lett.* **B396** (1997) 295.
 25. A. Breakstone et al., *Z. Phys.* **C48** (1990) 569.
 26. K.T. Chao, *Commun. Theor. Phys.* **27** (1997) 263.
 27. J.F. Bolzan, W.F. Palmer, S.S. Pinsky, *Phys. Rev.* **D15** (1977) 3460; *Phys. Rev.* **D14** (1976) 3202; *Phys. Rev.* **D14** (1976) 1920; W.F. Palmer, S.S. Pinsky, *Phys. Rev.* **D22** (1980) 223; *Phys. Rev.* **D14** (1976) 1916; J.F. Bolzan, K.A. Geer, W.F. Palmer, S.S. Pinsky, *Phys. Lett.* **B59** (1975) 351; *Phys. Rev. Lett.* **35** (1975) 419.
 28. N. Isgur, P. Geiger, *Phys. Rev.* **D47** (1993) 5050; N. Isgur, *Proc. of 25th INS Int. Symposium on Nuclear and Particle Physics with High-Intensity Proton Accelerators* (Dec. 1996, Tokyo, Japan) JLAB-THY-97-14.
 29. N.A. Törnqvist, *Phys. Rev. Lett.* **67** (1991) 556; C.B. Dover, T. Gutsche, A. Faessler, *Phys. Rev.* **C43** (1991) 379.
 30. T. Barnes, F.E. Close, P.R. Page, E.S. Swanson, *Phys. Rev.* **D55** (1997) 4157.
 31. K. Dooley, E.S. Swanson, T. Barnes, *Phys. Lett.* **B275** (1992) 478.
 32. J.Z. Bai et al. (BES Collab.) *Phys. Rev. Lett.* **77** (1996) 3959; Y.C. Zhu, *Proc. of Intersections between Particle and Nuclear Physics* (May 1997, Big Sky, Montana) p. 476, AIP conference proceedings 412, ed. T.W. Donnelly (AIP, New York, 1997).
 33. A.V. Anisovich et al., “Observation of $f_0(1770) \rightarrow \eta\eta$ ”; D.V. Bugg et al., *Phys. Lett.* **B353** (1995) 378; W. Dunwoodie, SLAC-PUB-7163; *Proc. of HADRON '97*, p. 753 [8]; G. Carlino (L3 Collab.) *Proc. of XVth Particle and Nuclei Int. Conf. (PANIC'99)* (June 1999, Uppsala, Sweden).
 34. M.M. Brisudova, L. Burakovsky, T. Goldman, hep-ph/9810296.
 35. R. Kokoski, N. Isgur, *Phys. Rev.* **D 35** (1987) 907.
 36. H.G. Blundell, S. Godfrey, *Phys. Rev.* **D53** (1996) 3700.
 37. E. Klempt, *Proc. of HADRON'97*, p. 867 [8].
 38. A. Palano, *Proc. of Workshop on Hadron Spectroscopy* [15].
 39. J. Cao, T. Huang, H.-F. Wu, *Phys. Rev.* **D57** (1998) 4154.
 40. Yu. S. Kalashnikova, Yu. B. Yufryakov, *Phys. Atom. Nucl.* **60** (1997) 307; *Phys. Lett.* **B359** (1995) 175; Yu. S. Kalashnikova, *Phys. Atom. Nucl.* **59** (1996) 1303; *Proc. of Int. Conf. on Hadron Spectroscopy (HADRON'95)* (July 1995, Manchester, U.K.) p. 533, eds. M.C. Birse et al. (World Scientific, Singapore, 1996) hep-ph/9509340.
 41. T. Manke, I.T. Drummond, R.R. Horgan, H.P. Shanahan, *Phys. Rev.* **D57** (1998) 3829.

A Appendix: 13×13 mass matrix

The 13×13 mass matrix for solution 2b is

$$\begin{pmatrix} 2.25 & 0.08 & 0.08 & \sqrt{2} & 0.08 & 0.08 & \sqrt{2} & 0.13 & 0.13 & \sqrt{2} & 0.14 & 0.14 & \sqrt{2} & 0.08 & 0.08 & \sqrt{2} & 0.12 & 0.12 & \sqrt{2} \\ 0.08 & 2.55 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.08 & \sqrt{2} & 0 & 2.35 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.08 & 0 & 0 & 0 & 2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.08 & \sqrt{2} & 0 & 0 & 0 & 2.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.13 & 0 & 0 & 0 & 0 & 0 & 2.27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.13 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 2.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.14 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.94 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.08 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.95 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.08 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.66 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.55 & 0 & 0 & 0 & 0 & 0 \\ 0.12 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.318 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.1})$$

The physical masses are

$$2.67, 2.53, 2.47, 2.34, 2.29, 2.23, 2.12, 2.01, 1.95, 1.81, 1.64, 1.52, 1.28, \quad (\text{A.2})$$

and the valence content of the physical states is

$$\begin{pmatrix} 0.65 & 0.45 & 0.23 & 0.32 & 0.20 & 0.21 & 0.19 & 0.18 & 0.18 & 0.07 & 0.07 & 0.07 & 0.08 \\ 0.19 & -0.84 & 0.12 & 0.46 & 0.09 & 0.09 & 0.07 & 0.07 & 0.06 & 0.02 & 0.02 & 0.02 & 0.03 \\ 0.27 & -0.29 & 0.25 & -0.82 & 0.18 & 0.18 & 0.12 & 0.12 & 0.10 & 0.04 & 0.04 & 0.04 & 0.04 \\ 0.11 & -0.04 & -0.88 & -0.05 & 0.37 & 0.23 & 0.07 & 0.09 & 0.06 & 0.02 & 0.02 & 0.02 & 0.02 \\ 0.08 & -0.02 & -0.15 & -0.03 & -0.79 & 0.57 & 0.06 & 0.08 & 0.05 & 0.02 & 0.01 & 0.01 & 0.01 \\ 0.23 & -0.06 & -0.21 & -0.07 & -0.36 & -0.69 & 0.24 & 0.43 & 0.16 & 0.07 & 0.05 & 0.04 & 0.04 \\ -0.17 & 0.03 & 0.08 & 0.04 & 0.11 & 0.15 & -0.44 & 0.83 & -0.19 & -0.08 & -0.04 & -0.04 & -0.04 \\ -0.18 & 0.03 & 0.06 & 0.03 & 0.07 & 0.09 & 0.74 & 0.17 & -0.54 & -0.26 & -0.06 & -0.05 & -0.04 \\ -0.02 & 0.00 & 0.00 & 0.00 & 0.00 & 0.01 & 0.03 & 0.01 & -0.38 & 0.92 & -0.01 & -0.00 & -0.00 \\ -0.43 & 0.05 & 0.09 & 0.05 & 0.10 & 0.12 & 0.32 & 0.17 & 0.64 & 0.24 & -0.33 & -0.20 & -0.15 \\ -0.20 & 0.02 & 0.03 & 0.02 & 0.03 & 0.04 & 0.09 & 0.05 & 0.13 & 0.05 & 0.92 & -0.28 & -0.10 \\ -0.22 & 0.02 & 0.03 & 0.02 & 0.03 & 0.04 & 0.08 & 0.05 & 0.10 & 0.04 & 0.18 & 0.93 & -0.18 \\ -0.23 & 0.01 & 0.02 & 0.01 & 0.03 & 0.03 & 0.05 & 0.04 & 0.07 & 0.03 & 0.07 & 0.10 & 0.96 \end{pmatrix}. \quad (\text{A.3})$$

Solution 2b is similar to the 9×9 solution 2 in (7)-(9) but is not the 13×13 counterpart of it, as evidenced by the different couplings of the common primitive states and the different primitive glueball mass.

We now study the stability of the solution under parameter changes. When we change the primitive $1F$ $n\bar{n}$ and $s\bar{s}$ masses upwards by 50 and 40 MeV respectively, reflecting our lack of knowledge of these parameters, the physical masses all remain consistent with experiment, except for $f_J(2220)$, which is 6 MeV higher than the experimental mean [1]. Valence contents of states far away in mass from $1F$ remain essentially constant. The largest changes are found for $f_2(2010)$, where valence contents less than 0.2 change on average by 70%. The dominant content changes by 1% and the next most dominant by

20%. A further change of the $2P$ $s\bar{s}$ mass downwards by 30 MeV yields the largest changes for the valence content of $f_2(1950)$, by similar amounts as before.

B Appendix: Motivation for mass matrix

If an appropriate basis is chosen, the hamiltonian for n isovector states up to a certain mass can be taken to be a diagonal $n \times n$ matrix $\mathcal{N} \equiv \text{diag}(N_1, N_2, \dots, N_n)$, where the entries are real and positive. These entries are identified with the masses of the physical states in the experimental isovector spectrum. Note that no assumption is made about the nature of the states, e.g. conventional or hybrid meson or four-quark state. In a world of only u, d quarks, this work assumes degeneracy of isovector and

isoscalar states, motivated in [10]. The hamiltonian for the isoscalar states is simply the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} G & \sqrt{2}g \\ \sqrt{2}g^T & \mathcal{N} \end{pmatrix} \quad (\text{B.4})$$

where a new state, which cannot exist in the isovector sector, the glueball, has been added. G is the primitive glueball mass, and g is a n -dimensional row of (real) couplings of either $u\bar{u}$ or $d\bar{d}$. These couplings are $\mathcal{O}(\frac{1}{\sqrt{N_c}})$ in the large number of colours N_c expansion of QCD. Of course, any number of extra glueballs can in principle be added [10].

If the strange quark is also incorporated, the isoscalar matrix becomes (1). Here the matrix element between $n\bar{n}$ states i and j ($i \neq j$) is zero because of the diagonality of \mathcal{N} in (B.4). The strange quark states are taken to be heavier analogues of the light ones, so that the mixing between $s\bar{s}$ states i and j ($i \neq j$) is zero, and the diagonal entries contain the masses S_i of the strange quark states. The remaining possible mixing is between $n\bar{n}$ state i and $s\bar{s}$ state j , which is $\mathcal{O}(\frac{1}{N_c})$ [11] and was found tiny in recent lattice QCD simulations [23], and is neglected in this work. The strange quark states are assumed to couple in the same way to the glueball as $u\bar{u}$ and $d\bar{d}$, which is the SU(3) limit. For ground state isoscalar scalar mixing, lattice QCD obtains the ratio of $u\bar{u}$ and $d\bar{d}$ to $s\bar{s}$ glueball coupling to be 1.198 ± 0.072 [23], while the SU(3) limit is 1.

C Appendix: Decay formalism

The mass matrix in 1 can be viewed as forming part of a hamiltonian H that describes an effective theory. This part of the hamiltonian is diagonalized to yield the physical states. One can then a posteriori add to the hamiltonian a part that describes coupling to a decay channel.

$$\begin{aligned} H = & g^* G g + \sum_{i=1}^n (s_i^* S_i s_i + n_i^* N_i n_i) + \left(\sum_{i=1}^n g_i g^* (s_i + \sqrt{2} n_i) \right. \\ & \left. + c.c. \right) + (\gamma^g g^* + \sum_{i=1}^n (\gamma_i^s s_i^* + \gamma_i^n n_i^*)) (BC) \end{aligned} \quad (\text{C.5})$$

where G, N_i and S_i are the masses and g_i the couplings in 1; g, n_i, s_i the corresponding primitive glueball, $n\bar{n}$ and $s\bar{s}$ meson fields; and $\gamma^g, \gamma_i^n, \gamma_i^s$ represent the couplings of the respective primitive states to the decay channel field (BC) . Spin indices for the primitive tensor states have been suppressed in H , e.g. $g^* G g$ stands for $g_{\mu\nu}^* G g^{\mu\nu}$, where $g^{\mu\nu}$ is a symmetric and traceless Lorentz tensor.

Now write the hamiltonian in shortened notation

$$\begin{aligned} H = & \mathbf{m}^\dagger \mathcal{M} \mathbf{m} + \mathbf{m}^\dagger \gamma (BC) & \mathbf{m} \equiv (g, s_1, n_1 \dots s_n, n_n)^T \\ \gamma \equiv & (\gamma^g, \gamma_1^s, \gamma_1^n \dots \gamma_n^s, \gamma_n^n)^T \end{aligned} \quad (\text{C.6})$$

where \mathcal{M} is defined as the matrix (1). Since \mathcal{M} is real and symmetric it can be diagonalized by use of the (real) orthogonal ($\Omega^{-1} = \Omega^T$) valence content matrix Ω , to yield the diagonal mass matrix of (real) eigenvalues (physical masses) $\mathcal{M}_D = \Omega \mathcal{M} \Omega^{-1}$. The eigenvectors (physical states) are $\tilde{\mathbf{m}} = \Omega \mathbf{m}$, where \mathbf{m} denotes the primitive states. The j^{th} row in Ω gives the valence content of the j^{th} physical state. Several practical examples of the diagonalization procedure can be found in this work. For example, for the \mathcal{M} in (4) \mathcal{M}_D is the matrix with diagonal entries (5) and Ω is (6).

The hamiltonian becomes

$$H = \tilde{\mathbf{m}}^\dagger \mathcal{M}_D \tilde{\mathbf{m}} + \tilde{\mathbf{m}}^\dagger \Omega \gamma (BC) \quad (\text{C.7})$$

The first term was discussed in detail in [27]. The second term shows clearly that in order to calculate the decay amplitude of a physical state to (BC) it is necessary to add the decay amplitudes of all its primitive components, weighted by their valence content. This was used, but not explicitly demonstrated in [27].